Appendix A

All Sky Imager Alignment

Three angular degrees of freedom are needed to align an All Sky Imager with respect to a fixed external co–ordinate system; two angles set the direction of the ASI optic axis and one determines a rotation about that axis. This section will define one such triplet of alignment angles and use stellar fields to determine their values.

We employ a fixed external Cartesian co–ordinate system as a reference frame for the rotations needed to define the orientation of the ASI, and choose the $\hat{z}$ axis to point to the zenith, the $\hat{x}$ axis to have azimuth $\phi = 0°$, and the $\hat{y}$ axis to have $\phi = 90°$. We denote the zenith angle by θ. By adopting the convention that positive rotations conform to a right–hand–rule, we can write matrices describing rotations around the three reference axes. The matrix, $R_z[\alpha]$, defining a rotation about $\hat{z}$ by some angle α is shown below. The corresponding matrices for $\hat{x}$ and $\hat{y}$ can be generated by cyclic permutations of $R_z[\alpha]$.

$$R_z[\alpha] = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

An arbitrarily oriented ASI can be obtained by applying three rotations to an ASI whose optic axis is originally along $\hat{z}$ (the zenith) and whose columns are originally aligned along $\hat{x}$ ($\phi = 0$). The rotations and their physical interpretation are summarized below. The order is important.

$R_z[\gamma]$	rotate the camera about the optic axis by γ
$R_y[\beta]$	rotate the optic axis to a zenith angle β
$R_z[\alpha]$	rotate the optic axis to an azimuthal angle α

The composite rotation matrix

$$R[\alpha, \beta, \gamma] = R_z[\alpha] R_y[\beta] R_z[\gamma]$$

defines the orientation of the camera co–ordinate system and has the additional property that when applied to the components of an arbitrary vector with respect to the **camera** or ASI co–ordinate system, yields the components of that vector with respect to the **fixed** co–ordinate system.

$$\bar{V}_{fixed} = R[\alpha, \beta, \gamma]\, \bar{V}_{asi}$$

The way is now clear to determine the triple (α, β, γ) of alignment angles:

1. image a star (or planet) in the field of view of the camera.
2. use an ephemeris to determine the **fixed** components of a vector to the star from its azimuth ϕ and zenith angle θ.
3. from the image of the star on the CCD detector determine the components of that same vector with respect to the **camera** frame of reference.
4. adjust the triplet of angles such that the above equation relating the **fixed and camera** co-ordinate systems is self consistent.

In general the above method will not permit a unique solution if but a single vector is used in the analysis. This restriction is easily overcome by repeating the process on a set of vectors generated by imaging several stars or planets, or by imaging the same object at different times during the night. A solution is then found by exhaustively searching the solution space of possible angles for the minimal least squares error over the complete set of vectors. The volume of the solution space can be reduced to manageable proportions by including constraints generated when an ASI is placed in the field; namely, the optic axis is generally known to be within a few degrees of the zenith, and the columns of the CCD are approximately aligned along the $\hat{x}$ axis.

Transformation Aside

To convert azimuth and zenith angles into Cartesian vector components with respect to the fixed frame, use:

$$\begin{aligned} x &= \sin(\theta)\cos(\phi) \\ y &= \sin(\theta)\sin(\phi) \\ z &= \cos(\theta) \end{aligned}$$

To convert the image co-ordinates (row, col) into a Cartesian vector pointing to the object being imaged, first find the azimuth and polar angles using:

$$\begin{aligned} \Delta y &= \text{col} - c_o \\ \Delta x &= \text{row} - r_o \\ r &= \sqrt{(\Delta x^2 + \Delta y^2)} \\ \theta &= k_1 * r + k_2 * r^2 \\ \phi &= \arctan(\Delta y / \Delta x) \end{aligned}$$

and then use the previous transform to convert to a Cartesian system. In the above equations, (r_o, c_o) is the origin of the optic axis on the CCD, and k_1 and k_2 are constants in the conversion between polar angle and radial distance.

Input Data

The input data for the calculation are presented in the table below where:

- **id** is an image reference number.
- Δx and Δy are the centroided row and column pixel co-ordinates of the image with respect to the intersection of the optic axis and the CCD.
- **r** is the radial distance to the image in pixels.
- ϕ and θ are the azimuthal and zenith angles calculated from an ephemeris.
- $\Delta\phi$ and $\Delta\theta$ are the respective differences between the azimuth and zenith angles when calculated from an ephemeris and from the image on the CCD. In particular:

$$\begin{aligned}\Delta\phi &= \phi_{ephem} - \phi_{asi} \\ \Delta\theta &= \theta_{ephem} - \theta_{asi}\end{aligned}$$

id	Δy	Δx	r	ϕ	$\Delta\phi$	θ	$\Delta\theta$
260	68.30	42.30	80.34	64.35	6.12	54.67	-1.17
262	-1.70	75.80	75.82	5.43	6.72	51.03	-1.23
263	20.30	103.20	105.18	17.67	6.54	74.37	-2.03
264	59.60	82.80	102.02	42.30	6.55	72.45	-1.25
265	42.30	68.30	80.34	38.00	6.23	54.28	-1.56
50	61.30	46.30	76.82	58.92	5.98	52.07	-0.99
52	-10.20	75.30	75.99	-1.35	6.36	50.93	-1.46
53	10.80	103.80	104.36	12.45	6.51	73.60	-2.10
54	51.10	86.50	100.47	37.15	6.58	70.58	-1.80
55	33.30	70.30	77.79	31.82	6.47	52.60	-1.22

Two observations are in order:

- The azimuthal angles determined from the ephemeris are systematically larger than the corresponding quantities determined from the image on the CCD by an amount on the order of 6.5°. We therefore expect the azimuthal correction to be of this magnitude.
- The systematically small zenith angle differences indicate that the optic axis is pointing close to the zenith.

ASI Orientation

The solution which minimized the residual errors for the above set of data was given by:

$$\begin{aligned} \alpha &= -160.4 \\ \beta &= +1.52 \\ \gamma &= +166.7 \\ (r_o,\ c_o) &= (127.7,\ 129.2) \\ (k_1,\ k_2) &= (0.593740,\ 0.001261) \end{aligned}$$

Because the optic axis is co-aligned with the zenith to within 1.5°, the rotations α and γ are essentially around the same axis and can be co–added to give a net rotation of 6.3°. This is the azimuthal misalignment expected. Because positive azimuth is a counterclockwise rotation about the zenith, the image on the CCD, to first order, must be rotated counterclockwise about the zenith by 6.3° to make it agree with the position calculated from the ephemeris. The exact transformation, which includes the 1.5° tilt of the optic axis away from the zenith, is given by applying the full rotation matrix $R[\alpha, \beta, \gamma]$ to each point in the image. In particular:

- for each pixel in the image, calculate the vector $\bar{V}_{asi}$ which points to the object being imaged. See the section on transformations for the equations necessary for this operation.
- apply $R[\alpha, \beta, \gamma]$ to this vector, thereby correcting for the orientation of the ASI.
- convert the corrected vector back into pixel co-ordinates or into azimuth and zenith angles. The relevant formulae are again in the section on transformations.

Applying the above procedure to the stellar images used in determining the orientation of the ASI gives the following table. The terms $\Delta\theta$ and $\Delta\phi$, which give the discrepancy between the measured and ephemeris values, have been significantly reduced in comparison with the corresponding values calculated for the original data.

id	Δy	Δx	r	ϕ	$\Delta\phi$	θ	$\Delta\theta$
260	71.73	33.07	78.99	64.35	-0.90	54.67	-0.10
262	6.03	73.69	73.94	5.43	0.76	51.03	0.24
263	30.89	98.68	103.40	17.67	0.28	74.37	-0.51
264	67.41	74.36	100.36	42.30	0.10	72.45	0.16
265	48.75	61.55	78.52	38.00	-0.38	54.28	-0.11
50	65.22	37.72	75.34	58.92	-1.04	52.07	0.18
52	-2.43	74.14	74.18	-1.35	0.53	50.93	-0.05
53	21.61	100.29	102.59	12.45	0.29	73.60	-0.59
54	59.42	78.87	98.75	37.15	0.16	70.58	-0.34
55	40.06	64.47	75.90	31.82	-0.04	52.60	0.27